

Muckenhoupt's (A_p) condition and the existence of the optimal martingale measure

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July 22, 2015

Abstract

In the problem of optimal investment with utility function defined on $(0, \infty)$, we formulate sufficient conditions for the dual optimizer to be a uniformly integrable martingale. Our key requirement consists of the existence of a martingale measure whose density process satisfies the probabilistic Muckenhoupt (A_p) condition for the power $p = 1/(1 - a)$, where $a \in (0, 1)$ is a lower bound on the relative risk-aversion of the utility function. We construct a counterexample showing that this (A_p) condition is sharp.

Keywords: utility maximization, optimal martingale measure, BMO martingales, (A_p) condition.

AMS Subject Classification (2010): 60G44, 91G10.

1 Introduction

An unpleasant qualitative feature of the general theory of optimal investment with a utility function defined on $(0, \infty)$ is that the dual optimizer \hat{Y} may not be a uniformly integrable martingale. In the presence of jumps, it may even fail to be a local martingale. The corresponding counterexamples can be found in [12]. In this paper, we seek to provide conditions under

*The author also holds a part-time position at the University of Oxford. This research was supported in part by the Oxford-Man Institute for Quantitative Finance at the University of Oxford.

which the uniform martingale property for \hat{Y} holds and thus, \hat{Y}/\hat{Y}_0 defines the density process of the *optimal* martingale measure $\hat{\mathbb{Q}}$.

The question of whether \hat{Y} is a uniformly integrable martingale is of longstanding interest in mathematical finance and can be traced back to [8] and [10]. This problem naturally arises in situations involving utility-based arguments. For instance, it is relevant for pricing in incomplete markets, where according to [9] the existence of $\hat{\mathbb{Q}}$ is equivalent to the uniqueness of marginal utility-based prices for every bounded contingent claim.

Our key requirement consists of the existence of a dual supermartingale Z , which satisfies the probabilistic Muckenhoupt (A_p) condition for the power $p > 1$ such that

$$(1.1) \quad p = \frac{1}{1-a}.$$

Here $a \in (0, 1)$ is a lower bound on the relative risk-aversion of the utility function. As we prove in Theorem 5.1, this condition, along with the existence of an upper bound for the relative risk-aversion, yields $(A_{p'})$ for \hat{Y} for some $p' > 1$. This property in turn implies that the dual minimizer \hat{Y} is of class **(D)**, that is, the family of its values evaluated at all stopping times is uniformly integrable. In Proposition 6.1, we construct a counterexample showing that the bound (1.1) is the best possible for \hat{Y} to be of class **(D)** even in the case of power utilities and continuous stock prices.

A similar idea of passing regularity from some dual element to the optimal one has been employed in [6], [7] and [2] for respectively, quadratic, power and exponential utility functions defined on the whole real line. These papers use appropriate versions of the Reverse Hölder (R_q) inequality which is dual to (A_p) . Note that contrary to (A_p) , the uniform integrability property is not implied but rather *required* by (R_q) . While this requirement is not a problem for real-line utilities, where the optimal martingale measures always exist, it is clearly an issue for utility functions defined on $(0, \infty)$.

Even if the dual minimizer \hat{Y} is of class **(D)**, it may not be a martingale, due to the lack of the local martingale property; see the single-period example for logarithmic utility in [12, Example 5.1']. In Proposition 4.2 we prove that every *maximal* dual supermartingale (in particular, \hat{Y}) is a local martingale if the ratio of any two positive wealth processes is σ -bounded.

Our main results, Theorems 5.1 and 5.3, are stated in Section 5. They are accompanied by Corollaries 5.5 and 5.6, which exploit well known connections between the (A_p) condition and BMO martingales.

2 Setup

We use the same framework as in [12, 13] and refer to these papers for more details. There is a financial market with a bank account paying zero interest and d stocks. The process of stocks' prices $S = (S^i)$ is a semimartingale with values in \mathbf{R}^d on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$. Here T is a finite maturity and $\mathcal{F} = \mathcal{F}_T$, but we remark that our results also hold for the case of infinite maturity.

A (self-financing) portfolio is defined by an initial capital $x \in \mathbf{R}$ and a predictable S -integrable process $H = (H^i)$ with values in \mathbf{R}^d of the number of stocks. Its corresponding wealth process X evolves as

$$X_t = x + \int_0^t H_u dS_u, \quad t \in [0, T].$$

We denote by \mathcal{X} the family of non-negative wealth processes:

$$\mathcal{X} \triangleq \{X \geq 0 : X \text{ is a wealth process}\}$$

and by \mathcal{Q} the family of equivalent local martingale measures for \mathcal{X} :

$$\mathcal{Q} \triangleq \{\mathbb{Q} \sim \mathbb{P} : \text{every } X \in \mathcal{X} \text{ is a local martingale under } \mathbb{Q}\}.$$

We assume that

$$(2.1) \quad \mathcal{Q} \neq \emptyset,$$

which is equivalent to the absence of arbitrage; see [3, 5].

There is an economic agent whose preferences over terminal wealth are modeled by a utility function U defined on $(0, \infty)$. We assume that U is of *Inada type*, that is, it is strictly concave, strictly increasing, continuously differentiable on $(0, \infty)$, and

$$U'(0) = \lim_{x \rightarrow 0} U'(x) = \infty, \quad U'(\infty) = \lim_{x \rightarrow \infty} U'(x) = 0.$$

For a given initial capital $x > 0$, the goal of the agent is to maximize the expected utility of terminal wealth. The value function of this problem is denoted by

$$(2.2) \quad u(x) = \sup_{X \in \mathcal{X}, X_0 = x} \mathbb{E}[U(X_T)].$$

Following [12], we define the dual optimization problem to (2.2) as

$$(2.3) \quad v(y) = \inf_{Y \in \mathcal{Y}, Y_0 = y} \mathbb{E}[V(Y_T)], \quad y > 0,$$

where V is the convex conjugate to U :

$$V(y) = \sup_{x > 0} \{U(x) - xy\}, \quad y > 0,$$

and \mathcal{Y} is the family of “dual” supermartingales to \mathcal{X} :

$$\mathcal{Y} = \{Y \geq 0 : XY \text{ is a supermartingale for every } X \in \mathcal{X}\}.$$

Note that the set \mathcal{Y} contains the density processes of all $\mathbb{Q} \in \mathcal{Q}$ and that, as $1 \in \mathcal{X}$, every element of \mathcal{Y} is a supermartingale.

It is known, see [13, Theorem 2], that under (2.1) and

$$(2.4) \quad v(y) < \infty, \quad y > 0,$$

the value functions u and $-v$ are of Inada type, v is the convex conjugate to u , and

$$(2.5) \quad v(y) = \inf_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E} \left[V \left(y \frac{d\mathbb{Q}}{d\mathbb{P}} \right) \right], \quad y > 0.$$

The solutions $X(x)$ to (2.2) and $Y(y)$ to (2.3) exist. If $y = u'(x)$ or, equivalently, $x = -v'(y)$, then

$$U'(X_T(x)) = Y_T(y),$$

and the product $X(x)Y(y)$ is a uniformly integrable martingale.

The last two properties actually characterize optimal $X(x)$ and $Y(y)$. For convenience of future references, we recall this “verification” result.

Lemma 2.1. *Let $\hat{X} \in \mathcal{X}$ and $\hat{Y} \in \mathcal{Y}$ be such that*

$$U'(\hat{X}_T) = \hat{Y}_T, \quad \mathbb{E} [V(\hat{Y}_T)] < \infty, \quad \mathbb{E} [\hat{X}_T \hat{Y}_T] = \hat{X}_0 \hat{Y}_0.$$

Then \hat{X} solves (2.2) for $x = \hat{X}_0$ and \hat{Y} solves (2.5) for $y = \hat{Y}_0$.

Proof. The result follows immediately from the identity

$$U(\hat{X}_T) = V(\hat{Y}_T) + \hat{X}_T \hat{Y}_T$$

and the inequalities

$$\begin{aligned} U(X_T) &\leq V(\hat{Y}_T) + X_T \hat{Y}_T, \quad X \in \mathcal{X}, \\ U(\hat{X}_T) &\leq V(Y_T) + \hat{X}_T Y_T, \quad Y \in \mathcal{Y}, \end{aligned}$$

after we recall that XY is a supermartingale for all $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. \square

The goal of the paper is to find sufficient conditions for the lower bound in (2.5) to be attained at some $\mathbb{Q}(y) \in \mathcal{Q}$ called the *optimal martingale measure* or, equivalently, for the dual minimizer $Y(y)$ to be a *uniformly integrable martingale*; in this case,

$$Y_T(y) = y \frac{d\mathbb{Q}(y)}{d\mathbb{P}}.$$

Our criteria are stated in Theorem 5.1 below, where a key role is played by the probabilistic version of the classical Muckenhoupt (A_p) condition.

3 (A_p) condition for the dual minimizer

Following [11, Section 2.3], we recall the probabilistic (A_p) condition.

Definition 3.1. Let $p > 1$. An optional process $R \geq 0$ satisfies (A_p) if $R_T > 0$ and there is a constant $C > 0$ such that for every stopping time τ

$$\mathbb{E} \left[\left(\frac{R_\tau}{R_T} \right)^{\frac{1}{p-1}} \middle| \mathcal{F}_\tau \right] \leq C.$$

Observe that if R satisfies (A_p) , then R satisfies $(A_{p'})$ for every $p' \geq p$.

An important consequence of the (A_p) condition is a uniform integrability property. For continuous local martingales this fact is well known and can be found e.g., in [11, Section 2.3].

Lemma 3.2. *If an optional process $R \geq 0$ satisfies (A_p) for some $p > 1$ and $\mathbb{E}[R_T] < \infty$, then R is of class (\mathbf{D}) :*

$\{R_\tau : \tau \text{ is a stopping time}\}$ is uniformly integrable.

Proof. Let τ be a stopping time. As $p > 1$, the function $x \mapsto x^{-\frac{1}{p-1}}$ is convex. Hence, by Jensen's inequality,

$$\mathbb{E} \left[\left(\frac{R_\tau}{R_T} \right)^{\frac{1}{p-1}} \middle| \mathcal{F}_\tau \right] = R_\tau^{\frac{1}{p-1}} \mathbb{E} \left[R_T^{-\frac{1}{p-1}} \middle| \mathcal{F}_\tau \right] \geq R_\tau^{\frac{1}{p-1}} (\mathbb{E}[R_T | \mathcal{F}_\tau])^{-\frac{1}{p-1}}.$$

Using the constant $C > 0$ from (A_p) , we obtain that

$$R_\tau \leq C^{p-1} \mathbb{E}[R_T | \mathcal{F}_\tau],$$

and the result follows. \square

To motivate the use of the (A_p) condition in the study of the dual minimizers $Y(y)$, $y > 0$, we first consider the case of power utility with a positive power.

Proposition 3.3. *Let (2.1) hold. Assume that*

$$U(x) = \frac{x^{1-a}}{1-a}, \quad x > 0,$$

with the relative risk-aversion $a \in (0, 1)$ and denote $p \triangleq \frac{1}{1-a} > 1$. Then for $y > 0$, the solution $Y(y)$ to the dual problem (2.3) exists if and only if

$$(3.1) \quad \mathbb{E} \left[Y_T^{-\frac{1}{p-1}} \right] < \infty \quad \text{for some } Y \in \mathcal{Y}$$

and, in this case, for every $Y \in \mathcal{Y}$, $Y > 0$ and every stopping time τ ,

$$\mathbb{E} \left[\left(\frac{Y_\tau(y)}{Y_T(y)} \right)^{\frac{1}{p-1}} \middle| \mathcal{F}_\tau \right] \leq \mathbb{E} \left[\left(\frac{Y_\tau}{Y_T} \right)^{\frac{1}{p-1}} \middle| \mathcal{F}_\tau \right].$$

In particular, $Y(y)$ satisfies (A_p) if and only if there is $Y \in \mathcal{Y}$ satisfying (A_p) .

Proof. Observe that the convex conjugate to U is given by

$$V(y) = \frac{a}{1-a} y^{-\frac{1-a}{a}} = (p-1) y^{-\frac{1}{p-1}}, \quad y > 0.$$

Then (3.1) is equivalent to (2.4), which, in turn, is equivalent to the existence of the optimal $Y(y)$, $y > 0$. Denote $\hat{Y} \triangleq Y(1)$. Clearly, $Y(y) = y\hat{Y}$.

Let a stopping time τ and a process $Y \in \mathcal{Y}$, $Y > 0$, be such that

$$\mathbb{E} \left[\left(\frac{Y_\tau}{Y_T} \right)^{\frac{1}{p-1}} \middle| \mathcal{F}_\tau \right] < \infty.$$

We have to show that

$$\xi \triangleq \mathbb{E} \left[\left(\frac{\hat{Y}_\tau}{\hat{Y}_T} \right)^{\frac{1}{p-1}} \middle| \mathcal{F}_\tau \right] - \mathbb{E} \left[\left(\frac{Y_\tau}{Y_T} \right)^{\frac{1}{p-1}} \middle| \mathcal{F}_\tau \right] \leq 0.$$

For a set $A \in \mathcal{F}_\tau$, the process

$$Z_t \triangleq \hat{Y}_t 1_{\{t \leq \tau\}} + \hat{Y}_\tau \left(\frac{Y_t}{Y_\tau} 1_A + \frac{\hat{Y}_t}{\hat{Y}_\tau} (1 - 1_A) \right) 1_{\{t > \tau\}}, \quad t \in [0, T],$$

belongs to \mathcal{Y} and is such that $Z_0 = 1$ and $Z_\tau = \widehat{Y}_\tau$. We obtain that

$$\begin{aligned}\mathbb{E} \left[\left(\frac{Z_\tau}{Z_T} \right)^{\frac{1}{p-1}} \middle| \mathcal{F}_\tau \right] &= \mathbb{E} \left[\left(\frac{Y_\tau}{Y_T} \right)^{\frac{1}{p-1}} \middle| \mathcal{F}_\tau \right] 1_A + \mathbb{E} \left[\left(\frac{\widehat{Y}_\tau}{\widehat{Y}_T} \right)^{\frac{1}{p-1}} \middle| \mathcal{F}_\tau \right] (1 - 1_A) \\ &= \mathbb{E} \left[\left(\frac{\widehat{Y}_\tau}{\widehat{Y}_T} \right)^{\frac{1}{p-1}} \middle| \mathcal{F}_\tau \right] - \xi 1_A.\end{aligned}$$

Dividing both sides by $Z_\tau^{\frac{1}{p-1}} = \widehat{Y}_\tau^{\frac{1}{p-1}}$ and choosing $A = \{\xi \geq 0\}$, we deduce that

$$\mathbb{E} \left[\left(\frac{1}{Z_T} \right)^{\frac{1}{p-1}} \right] = \mathbb{E} \left[\left(\frac{1}{\widehat{Y}_T} \right)^{\frac{1}{p-1}} \right] - \mathbb{E} \left[\left(\frac{1}{\widehat{Y}_T} \right)^{\frac{1}{p-1}} \max(\xi, 0) \right].$$

However, the optimality of $\widehat{Y} = Y(1)$ implies that

$$\mathbb{E} \left[\left(\frac{1}{\widehat{Y}_T} \right)^{\frac{1}{p-1}} \right] \leq \mathbb{E} \left[\left(\frac{1}{Z_T} \right)^{\frac{1}{p-1}} \right].$$

Hence $\xi \leq 0$. □

We now state the main result of the section.

Theorem 3.4. *Let (2.1) hold. Suppose that there are constants $0 < a < 1$, $b \geq a$ and $C > 0$ such that*

$$(3.2) \quad \frac{1}{C} \left(\frac{y}{x} \right)^a \leq \frac{U'(x)}{U'(y)} \leq C \left(\frac{y}{x} \right)^b, \quad x \leq y,$$

and there is a supermartingale $Z \in \mathcal{Y}$ satisfying (A_p) with

$$p = \frac{1}{1-a}.$$

Then for every $y > 0$, the solution $Y(y)$ to (2.3) exists and satisfies $(A_{p'})$ with

$$p' = 1 + \frac{b}{1-a}.$$

Remark 3.5. Notice that if the relative risk-aversion of U is well-defined and bounded away from 0 and ∞ , then in (3.2) we can take $C = 1$ and choose a and b as lower and upper bounds:

$$0 < a \leq -\frac{xU''(x)}{U'(x)} \leq b < \infty, \quad x > 0.$$

In particular, if

$$1 \leq -\frac{xU''(x)}{U'(x)} \leq b, \quad x > 0,$$

then choosing $a \in (0, 1)$ sufficiently close to 1 we fulfill the conditions of Theorem 3.4 if there exists a supermartingale $Z \in \mathcal{Y}$ satisfying (A_p) for some $p > 1$.

Observe also that for the positive power utility function U with relative risk-aversion $a \in (0, 1)$ we can select $b = a$ and then obtain same estimate as in Proposition 3.3:

$$p' = 1 + \frac{a}{1-a} = \frac{1}{1-a} = p.$$

The proof of Theorem 3.4 relies on the following lemma.

Lemma 3.6. *Assume (2.1) and suppose that there are constants $0 < a < 1$ and $C_1 > 0$ such that*

$$(3.3) \quad \frac{1}{C_1} \left(\frac{y}{x}\right)^a \leq \frac{U'(x)}{U'(y)}, \quad x \leq y,$$

and there is a supermartingale $Z \in \mathcal{Y}$ satisfying (A_p) with

$$p = \frac{1}{1-a}.$$

Then for every $y > 0$ the solution $Y(y)$ to (2.3) exists, and there is a constant $C_2 > 0$ such that for every stopping time τ and every $y > 0$,

$$(3.4) \quad \mathbb{E} [I(Y_T(y))Y_T(y) | \mathcal{F}_\tau] \leq C_2 I(Y_\tau(y))Y_\tau(y),$$

where $I = -V'$.

Remark 3.7. Recall that for $x = -v'(y)$ the optimal wealth process $X(x)$ has the terminal value

$$X_T(x) = -V'(Y_T(y)) = I(Y_T(y))$$

and the product $X(x)Y(y)$ is a uniformly integrable martingale. It follows that for every stopping time τ

$$X_\tau(x) = \frac{1}{Y_\tau(y)} \mathbb{E} [I(Y_T(y))Y_T(y) | \mathcal{F}_\tau]$$

and therefore, inequality (3.4) is equivalent to

$$X_\tau(x) \leq C_2 I(Y_\tau(y)).$$

Proof of Lemma 3.6. To show the existence of $Y(y)$ we need to verify (2.4). As $I = -V'$ is the inverse function to U , condition (3.3) is equivalent to

$$(3.5) \quad \frac{I(x)}{I(y)} \leq C_3 \left(\frac{y}{x} \right)^{1/a}, \quad x \leq y,$$

where $C_3 = C_1^{1/a}$. From (3.5) we deduce that for $y \leq 1$

$$\begin{aligned} V(y) &= V(1) + \int_y^1 I(t) dt \leq V(1) + C_3 I(1) \int_y^1 t^{-1/a} dt \\ &= V(1) + C_3 I(1) \frac{a}{1-a} (y^{-\frac{1-a}{a}} - 1) \\ &= V(1) + C_3 I(1) (p-1) (y^{-\frac{1}{p-1}} - 1). \end{aligned}$$

Hence, there is a constant $C_4 > 0$ such that

$$V(y) \leq C_4 (1 + y^{-\frac{1}{p-1}}), \quad y > 0.$$

As Z satisfies (A_p) , we have

$$\mathbb{E} \left[Z_T^{-\frac{1}{p-1}} \right] < \infty.$$

It follows that

$$v(y) \leq \mathbb{E} [V(yZ_T/Z_0)] < \infty, \quad y > 0,$$

which completes the proof of the existence of $Y(y)$.

Let τ be a stopping time and let $y > 0$. We set $\hat{Y} \triangleq Y(y)$ and define the process

$$Y_t \triangleq \hat{Y}_t 1_{\{t \leq \tau\}} + \hat{Y}_\tau \frac{Z_t}{Z_\tau} 1_{\{t > \tau\}}, \quad t \in [0, T].$$

Clearly, $Y \in \mathcal{Y}$ and $Y_0 = \hat{Y}_0 = y$. We represent

$$I(\hat{Y}_T) \hat{Y}_T = \xi_1 + \xi_2 + \xi_3,$$

by multiplying the left-side on the elements of the unity decomposition:

$$1 = 1_{\{\hat{Y}_\tau \leq \hat{Y}_T\}} + 1_{\{Y_T \leq \hat{Y}_T < \hat{Y}_\tau\}} + 1_{\{\hat{Y}_T < Y_T, \hat{Y}_T < \hat{Y}_\tau\}}.$$

For the first term, since $I = -V'$ is a decreasing function, we have that

$$\xi_1 = I(\hat{Y}_T) \hat{Y}_T 1_{\{\hat{Y}_\tau \leq \hat{Y}_T\}} \leq I(\hat{Y}_\tau) \hat{Y}_T.$$

Using the supermartingale property of \widehat{Y} , we obtain that

$$\mathbb{E}[\xi_1 | \mathcal{F}_\tau] \leq I(\widehat{Y}_\tau) \widehat{Y}_\tau.$$

For the second term, we deduce from (3.5) that

$$\begin{aligned} \xi_2 &= I(\widehat{Y}_T) \widehat{Y}_T 1_{\{Y_T \leq \widehat{Y}_T < \widehat{Y}_\tau\}} = I(\widehat{Y}_T) \widehat{Y}_T^{\frac{1}{a}} \widehat{Y}_T^{-\frac{1-a}{a}} 1_{\{Y_T \leq \widehat{Y}_T < \widehat{Y}_\tau\}} \\ &\leq C_3 I(\widehat{Y}_\tau) \widehat{Y}_\tau^{\frac{1}{a}} Y_T^{-\frac{1-a}{a}} = C_3 I(\widehat{Y}_\tau) \widehat{Y}_\tau \left(\frac{Z_\tau}{Z_T} \right)^{\frac{1-a}{a}} = C_3 I(\widehat{Y}_\tau) \widehat{Y}_\tau \left(\frac{Z_\tau}{Z_T} \right)^{\frac{1}{p-1}} \end{aligned}$$

and the (A_p) condition for Z yields the existence of a constant $C_5 > 0$ such that

$$\mathbb{E}[\xi_2 | \mathcal{F}_\tau] \leq C_5 I(\widehat{Y}_\tau) \widehat{Y}_\tau.$$

For the third term, we deduce from (3.5) that

$$\begin{aligned} \xi_3 &= I(\widehat{Y}_T) \widehat{Y}_T 1_{\{\widehat{Y}_T < Y_T, \widehat{Y}_T < \widehat{Y}_\tau\}} \leq I(\widehat{Y}_T) \widehat{Y}_T 1_{\{\widehat{Y}_T < Y_T\}} \\ &= I(\widehat{Y}_T)^a \widehat{Y}_T I(\widehat{Y}_T)^{1-a} 1_{\{\widehat{Y}_T < Y_T\}} \leq C_1 I(Y_T)^a Y_T I(\widehat{Y}_T)^{1-a} \\ &= C_1 (I(Y_T) Y_T)^a (I(\widehat{Y}_T) Y_T)^{1-a} \end{aligned}$$

and then from Hölder's inequality that

$$\mathbb{E}[\xi_3 | \mathcal{F}_\tau] \leq C_1 (\mathbb{E}[I(Y_T) Y_T | \mathcal{F}_\tau])^a \left(\mathbb{E}[I(\widehat{Y}_T) Y_T | \mathcal{F}_\tau] \right)^{1-a}.$$

We recall that the terminal wealth of the optimal investment strategy with $\widehat{X}_0 = -v'(y)$ is given by

$$I(\widehat{Y}_T) = \widehat{X}_T.$$

It follows that

$$\begin{aligned} \mathbb{E}[I(\widehat{Y}_T) Y_T | \mathcal{F}_\tau] &= \mathbb{E}[\widehat{X}_T Y_T | \mathcal{F}_\tau] \leq \widehat{X}_\tau Y_\tau = \widehat{X}_\tau \widehat{Y}_\tau \\ &= \mathbb{E}[\widehat{X}_T \widehat{Y}_T | \mathcal{F}_\tau] = \mathbb{E}[I(\widehat{Y}_T) \widehat{Y}_T | \mathcal{F}_\tau]. \end{aligned}$$

To estimate $\mathbb{E}[I(Y_T) Y_T | \mathcal{F}_\tau]$ we decompose

$$I(Y_T) Y_T = I(Y_T) Y_T 1_{\{\widehat{Y}_\tau \leq Y_T\}} + I(Y_T) Y_T 1_{\{\widehat{Y}_\tau > Y_T\}}.$$

Since I is decreasing, we have that

$$I(Y_T) Y_T 1_{\{\widehat{Y}_\tau \leq Y_T\}} \leq I(\widehat{Y}_\tau) Y_T.$$

As Y is a supermartingale and $Y_\tau = \widehat{Y}_\tau$, we obtain that

$$\mathbb{E} \left[I(Y_T) Y_T 1_{\{\widehat{Y}_\tau \leq Y_T\}} \middle| \mathcal{F}_\tau \right] \leq I(\widehat{Y}_\tau) Y_\tau = I(\widehat{Y}_\tau) \widehat{Y}_\tau.$$

For the second term, using (3.5) we deduce that

$$\begin{aligned} I(Y_T) Y_T 1_{\{\widehat{Y}_\tau > Y_T\}} &= I(Y_T) Y_T^{\frac{1}{a}} Y_T^{-\frac{1-a}{a}} 1_{\{\widehat{Y}_\tau > Y_T\}} \leq C_3 I(\widehat{Y}_\tau) \widehat{Y}_\tau^{\frac{1}{a}} Y_T^{-\frac{1-a}{a}} \\ &= C_3 I(\widehat{Y}_\tau) \widehat{Y}_\tau \left(\frac{\widehat{Y}_\tau}{Y_T} \right)^{\frac{1-a}{a}} = C_3 I(\widehat{Y}_\tau) \widehat{Y}_\tau \left(\frac{Z_\tau}{Z_T} \right)^{\frac{1}{p-1}} \end{aligned}$$

and the (A_p) condition for Z implies that

$$\mathbb{E} \left[I(Y_T) Y_T 1_{\{\widehat{Y}_\tau > Y_T\}} \middle| \mathcal{F}_\tau \right] \leq C_5 I(\widehat{Y}_\tau) \widehat{Y}_\tau.$$

Thus we have

$$\mathbb{E} [I(Y_T) Y_T | \mathcal{F}_\tau] \leq \eta \triangleq (1 + C_5) I(\widehat{Y}_\tau) \widehat{Y}_\tau$$

and then

$$\mathbb{E} [\xi_3 | \mathcal{F}_\tau] \leq C_1 \eta^a \left(\mathbb{E} [I(\widehat{Y}_T) \widehat{Y}_T | \mathcal{F}_\tau] \right)^{1-a}.$$

Adding together the estimates for $\mathbb{E} [\xi_i | \mathcal{F}_\tau]$ we obtain that

$$\mathbb{E} [I(\widehat{Y}_T) \widehat{Y}_T | \mathcal{F}_\tau] \leq \eta + C_1 \eta^a \left(\mathbb{E} [I(\widehat{Y}_T) \widehat{Y}_T | \mathcal{F}_\tau] \right)^{1-a}.$$

It follows that

$$\mathbb{E} [I(\widehat{Y}_T) \widehat{Y}_T | \mathcal{F}_\tau] \leq x^* \eta = x^* (1 + C_5) I(\widehat{Y}_\tau) \widehat{Y}_\tau,$$

where x^* is the root of

$$x = 1 + C_1 x^{1-a}, \quad x > 0.$$

We thus have proved inequality (3.4) with $C_2 = (1 + C_5) x^*$. \square

Proof of Theorem 3.4. Fix $y > 0$. In view of Lemma 3.6, we only have to verify that $\widehat{Y} \triangleq Y(y)$ satisfies $(A_{p'})$.

Denote $\widehat{X} \triangleq X(-v'(y))$ and recall that by Lemma 3.6 and Remark 3.7, there is $C_2 > 0$ such that, for every stopping time τ ,

$$\widehat{X}_\tau \leq C_2 I(\widehat{Y}_\tau).$$

Observe also that as $I = -V'$ is the inverse function to U' , the second inequality in (3.2) is equivalent to

$$\frac{y}{x} \leq C \left(\frac{I(x)}{I(y)} \right)^b, \quad x \leq y.$$

We fix a stopping time τ . Since $I(\widehat{Y}_T) = \widehat{X}_T$, we deduce from the inequalities above that

$$\left(\frac{\widehat{Y}_\tau}{\widehat{Y}_T} \right)^{1/b} \leq \max \left(1, C^{1/b} \frac{I(\widehat{Y}_T)}{I(\widehat{Y}_\tau)} \right) \leq \max \left(1, C_3 \frac{\widehat{X}_T}{\widehat{X}_\tau} \right),$$

where $C_3 = C^{1/b} C_2$. It follows that

$$\begin{aligned} \left(\frac{\widehat{Y}_\tau}{\widehat{Y}_T} \right)^{\frac{1}{p'-1}} &= \left(\frac{\widehat{Y}_\tau}{\widehat{Y}_T} \right)^{\frac{1-a}{b}} \leq \max \left(1, C_3^{1-a} \left(\frac{\widehat{X}_T}{\widehat{X}_\tau} \right)^{1-a} \right) \\ &\leq 1 + C_3^{1-a} \left(\frac{\widehat{X}_T Z_T}{\widehat{X}_\tau Z_\tau} \right)^{1-a} \left(\frac{Z_\tau}{Z_T} \right)^{1-a}. \end{aligned}$$

Denoting by $C_1 > 0$ the constant in the (A_p) condition for Z , we deduce from Hölder's inequality and the supermartingale property of $\widehat{X}Z$ that

$$\begin{aligned} \mathbb{E} \left[\left(\frac{\widehat{Y}_\tau}{\widehat{Y}_T} \right)^{\frac{1}{p'-1}} \middle| \mathcal{F}_\tau \right] &\leq 1 + C_3^{1-a} \left(\mathbb{E} \left[\frac{\widehat{X}_T Z_T}{\widehat{X}_\tau Z_\tau} \middle| \mathcal{F}_\tau \right] \right)^{1-a} \left(\mathbb{E} \left[\left(\frac{Z_\tau}{Z_T} \right)^{\frac{1}{p-1}} \middle| \mathcal{F}_\tau \right] \right)^a \\ &\leq 1 + C_3^{1-a} C_1^a. \end{aligned}$$

Hence, \widehat{Y} satisfies $(A_{p'})$. \square

4 Local martingale property for maximal elements of \mathcal{Y}

Even if the dual minimizer $Y(y)$ is uniformly integrable, it may not be a martingale, due to the lack of the local martingale property; see the single-period example for logarithmic utility in [12, Example 5.1']. Proposition 4.2 below yields sufficient conditions for every *maximal* element of \mathcal{Y} (in particular, for $Y(y)$) to be a local martingale.

A semimartingale R is called σ -bounded if there is a predictable process $h > 0$ such that the stochastic integral $\int h dR$ is bounded. Following [14], we make the following assumption.

Assumption 4.1. For all X and X' in \mathcal{X} such that $X > 0$, the process X'/X is σ -bounded.

Assumption 4.1 holds easily if stock price S is continuous. Theorem 3 in Appendix of [14] provides a sufficient condition in the presence of jumps. It states that *every* semimartingale R is σ -bounded if there is a finite-dimensional local martingale M such that every bounded purely discontinuous martingale N is a stochastic integral with respect to M .

Proposition 4.2. *Suppose that Assumption 4.1 holds. Let $Y \in \mathcal{Y}$ be such that YX' is a local martingale for some $X' \in \mathcal{X}$, $X' > 0$. Then YX is a local martingale for every $X \in \mathcal{X}$. In particular, Y is a local martingale.*

Proof. We assume first that $X' = Y = 1$. Let $X \in \mathcal{X}$. As X is σ -bounded, there is a predictable $h > 0$ such that

$$\left| \int h dX \right| \leq 1.$$

Since the bounded non-negative processes $1 \pm \int h dX$ belong to \mathcal{X} , they are supermartingales, which is only possible if $\int h dX$ is a martingale. It follows that X is a non-negative stochastic integral with respect to a martingale:

$$X = X_0 + \int \frac{1}{h} d\left(\int h dX\right) \geq 0.$$

Therefore, X is a local martingale, see [1]. Under the condition $X' = Y = 1$, the proof is obtained.

We now consider the general case. Without loss of generality, we can assume that $X'_0 = Y_0 = 1$. By localization, we can also assume that the local martingale YX' is uniformly integrable and then define a probability measure \mathbb{Q} with the density

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = X'_T Y_T.$$

Let $X \in \mathcal{X}$. We have that XY is a local martingale under \mathbb{P} if and only if X/X' is a local martingale under \mathbb{Q} .

By Assumption 4.1, the process X/X' is σ -bounded. Elementary computations show that X/X' is a wealth process in the financial market with stock price

$$S' = \left(\frac{1}{X'}, \frac{S}{X'} \right);$$

see [4]. The result now follows by applying the previous argument to the S' -market whose reference probability measure is given by \mathbb{Q} . \square

5 Existence of the optimal martingale measure

Recall that $X(x)$ denotes the optimal wealth process for the primal problem (2.2), while $Y(y)$ stands for the minimizer to the dual problem (2.3). As usual, the *density process* of a probability measure $\mathbb{R} \ll \mathbb{P}$ is a uniformly integrable martingale (under \mathbb{P}) with the terminal value $\frac{d\mathbb{R}}{d\mathbb{P}}$.

The following is the main result of the paper.

Theorem 5.1. *Let Assumption 4.1 hold. Suppose that there are constants $0 < a < 1$, $b \geq a$ and $C > 0$ such that*

$$(5.1) \quad \frac{1}{C} \left(\frac{y}{x}\right)^a \leq \frac{U'(x)}{U'(y)} \leq C \left(\frac{y}{x}\right)^b, \quad x \leq y,$$

and there is a martingale measure $\mathbb{Q} \in \mathcal{Q}$ whose density process Z satisfies (A_p) with

$$(5.2) \quad p = \frac{1}{1-a}.$$

Then for every $y > 0$ the optimal martingale measure $\mathbb{Q}(y)$ exists and its density process $Y(y)/y$ satisfies $(A_{p'})$ with

$$p' = 1 + \frac{b}{1-a}.$$

Proof. From Theorem 3.4 we obtain that the dual minimizer $Y(y)$ exists and satisfies $(A_{p'})$ and then from Lemma 3.2 that it is of class **(D)**. The local martingale property of $Y(y)$ follows from Proposition 4.2, if we account for Assumption 4.1 and the martingale property of $X(-v'(y))Y(y)$. Thus, $Y(y)$ is a uniformly integrable martingale and hence, $Y(y)/y$ is the density process of the optimal martingale measure $\mathbb{Q}(y)$. \square

We refer the reader to Remark 3.5 for a discussion of the conditions of Theorem 5.1.

Example 5.2. In a typical situation, the role of the “testing” martingale measure \mathbb{Q} is played by the *minimal* martingale measure, that is, by the optimal martingale measure for logarithmic utility. For a model of stock prices driven by a Brownian motion, its density process Z has the form:

$$Z_t = \mathcal{E}(-\lambda \cdot B)_t := \exp\left(-\int_0^t \lambda dB - \frac{1}{2} \int_0^t |\lambda_s|^2 ds\right), \quad t \in [0, T],$$

where B is an N -dimensional Brownian motion and λ is a predictable N -dimensional process of the *market price of risk*. We readily deduce that Z satisfies (A_p) for all $p > 1$ if both λ and the maturity T are bounded. This fact implies the assertions of Theorem 5.1, provided that inequalities (5.1) hold for some $a \in (0, 1)$, $b \geq a$ and $C > 0$ or, in particular, if the relative risk-aversion of U is bounded away from 0 and ∞ .

The following result shows that the key bound (5.2) is the best possible.

Theorem 5.3. *Let constants a and p be such that*

$$0 < a < 1 \quad \text{and} \quad p > \frac{1}{1-a}.$$

Then there exists a financial market with a continuous stock price S such that

1. *There is a $\mathbb{Q} \in \mathcal{Q}$ whose density process Z satisfies (A_p) .*
2. *In the optimal investment problem with the power utility function*

$$U(x) = \frac{x^{1-a}}{1-a}, \quad x > 0,$$

the dual minimizers $Y(y) = y\hat{Y}$, $y > 0$, are well-defined, but are not uniformly integrable martingales. In particular, the optimal martingale measure $\hat{\mathbb{Q}} = \mathbb{Q}(y)$ does not exist.

The proof of Theorem 5.3 follows from Proposition 6.1 below, which contains an exact counterexample.

We conclude the section with a couple of corollaries of Theorem 5.1 which exploit connections between the (A_p) condition and BMO martingales. Hereafter, we shall refer to [11] and therefore, restrict ourselves to the continuous case.

Assumption 5.4. All local martingales on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ are continuous.

From Assumption 5.4 we deduce that the density process of every $\mathbb{Q} \in \mathcal{Q}$ is a continuous uniformly integrable martingale and that the dual minimizer $Y(y)$ is a continuous local martingale.

We recall that a continuous local martingale M with $M_0 = 0$ belongs to BMO if there is a constant $C > 0$ such that

$$(5.3) \quad \mathbb{E} [\langle M \rangle_T - \langle M \rangle_\tau | \mathcal{F}_\tau] \leq C \quad \text{for every stopping time } \tau,$$

where $\langle M \rangle$ is the quadratic variation process for M . It is known that BMO is a Banach space with the norm

$$\|M\|_{\text{BMO}} \triangleq \inf \left\{ \sqrt{C} > 0 : (5.3) \text{ holds for } C > 0 \right\}.$$

We also recall that for a continuous local martingale M with $M_0 = 0$,

- (i) The stochastic exponential $\mathcal{E}(M) \triangleq e^{M - \langle M \rangle / 2}$ satisfies (A_p) for *some* $p > 1$ if and only if $M \in \text{BMO}$; see Theorem 2.4 in [11].
- (ii) The stochastic exponentials $\mathcal{E}(M)$ and $\mathcal{E}(-M)$ satisfy (A_p) for *all* $p > 1$ if and only if the martingale

$$(5.4) \quad q(M)_t \triangleq \mathbb{E} [\langle M \rangle_T | \mathcal{F}_t] - \mathbb{E} [\langle M \rangle_T], \quad t \in [0, T],$$

is well-defined and belongs to the closure in $\|\cdot\|_{\text{BMO}}$ of the space of bounded martingales; see Theorem 3.12 in [11].

Corollary 5.5. *Let Assumption 5.4 hold. Suppose that there are constants $b \geq 1$ and $C > 0$ such that*

$$(5.5) \quad \frac{1}{C} \left(\frac{y}{x} \right) \leq \frac{U'(x)}{U'(y)} \leq C \left(\frac{y}{x} \right)^b, \quad x \leq y,$$

and there is a martingale measure $\mathbb{Q} \in \mathcal{Q}$ with density process $Z = \mathcal{E}(M)$ with $M \in \text{BMO}$. Then for every $y > 0$ the optimal martingale measure $\mathbb{Q}(y)$ exists and its density process is given by $Y(y)/y = \mathcal{E}(M(y))$ with $M(y) \in \text{BMO}$.

Proof. From (i) we deduce that Z satisfies (A_p) for some $p > 1$. Clearly, (5.5) implies (5.1) for every $a \in (0, 1)$ and in particular for a satisfying (5.2). Theorem 5.1 then implies that $Y(y)/y$ satisfies $(A_{p'})$ for some $p' > 1$ and another application of (i) yields the result. \square

We notice that by (i) and Theorem 5.3 the power 1 in the first inequality of (5.5) cannot be replaced with any $a \in (0, 1)$, in order to guarantee that the optimal martingale measure $\mathbb{Q}(y)$ exists.

Corollary 5.6. *Let Assumption 5.4 hold and let inequality (5.1) be satisfied for some constants $0 < a < 1$, $b \geq a$ and $C > 0$. Suppose also that there is a martingale measure $\mathbb{Q} \in \mathcal{Q}$ whose density process $Z = \mathcal{E}(M)$ is such that the martingale $q(M)$ in (5.4) is well-defined and belongs to the closure in $\|\cdot\|_{\text{BMO}}$ of the space of bounded martingales. Then for every $y > 0$ the optimal martingale measure $\mathbb{Q}(y)$ exists and its density process is given by $Y(y)/y = \mathcal{E}(M(y))$ with $M(y) \in \text{BMO}$.*

Proof. The result follows directly from (ii) and Theorem 5.1. \square

6 Counterexample

In this section we construct an example of financial market satisfying the conditions of Theorem 5.3. For a semimartingale R , we denote by $\mathcal{E}(R)$ its stochastic exponential, that is, the solution of the linear equation:

$$d\mathcal{E}(R) = \mathcal{E}(R)_- dR, \quad \mathcal{E}(R)_0 = 1.$$

We start with an auxiliary filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{Q})$, which supports a Brownian motion $B = (B_t)$ and a counting process $N = (N_t)$ with the stochastic intensity $\lambda = (\lambda_t)$ given in (6.3) below; $B_0 = N_0 = 0$. We define the process

$$S_t \triangleq \mathcal{E}(B)_t = e^{B_t - t/2}, \quad t \geq 0,$$

and the stopping times

$$\begin{aligned} T_1 &\triangleq \inf \{t \geq 0 : S_t = 2\}, \\ T_2 &\triangleq \inf \{t \geq 0 : N_t = 1\}, \\ T &\triangleq T_1 \wedge T_2 = \min(T_1, T_2). \end{aligned}$$

We fix constants a and p such that

$$(6.1) \quad 0 < a < 1 \quad \text{and} \quad p > \frac{1}{1-a}$$

and choose a constant b such that

$$(6.2) \quad a < b < \frac{1}{q} \quad \text{and} \quad \gamma \leq \frac{1}{2}\delta(1-\delta),$$

where

$$\begin{aligned} q &\triangleq \frac{p}{p-1} < \frac{1}{a}, \\ \delta &\triangleq b - a > 0, \\ \gamma &\triangleq \frac{b}{2}(1 - qb) > 0. \end{aligned}$$

With this notation, we define the stochastic intensity $\lambda = (\lambda_t)$ as

$$(6.3) \quad \lambda_t \triangleq \frac{\gamma}{1 - (S_t/2)^\delta} 1_{\{t < T_1\}} + \gamma 1_{\{t \geq T_1\}}, \quad t \geq 0.$$

Recall that $N - \int \lambda dt$ is a local martingale under \mathbb{Q} .

Finally, we introduce a probability measure $\mathbb{P} \ll \mathbb{Q}$ with the density

$$\frac{d\mathbb{P}}{d\mathbb{Q}} = \frac{1}{\mathbb{E}^{\mathbb{Q}}[S_T^b]} S_T^b.$$

Notice that

$$(6.4) \quad \left\{ \frac{d\mathbb{P}}{d\mathbb{Q}} = 0 \right\} = \{S_T = 0\} = \{\mathcal{E}(B)_T = 0\} = \{T = \infty\}$$

and therefore, the stopping time T is finite under \mathbb{P} :

$$\mathbb{P}(T < \infty) = 1.$$

Proposition 6.1. *Assume (6.1) and (6.2) and consider the financial market with the price process S and the maturity T defined on the filtered probability space $(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$. Then*

1. *The probability measure \mathbb{Q} belongs to \mathcal{Q} and the density process Z of \mathbb{Q} with respect to \mathbb{P} satisfies (A_p) .*
2. *In the optimal investment problem with the power utility function*

$$(6.5) \quad U(x) = \frac{x^{1-a}}{1-a}, \quad x > 0,$$

the dual minimizers $Y(y) = y\hat{Y}$, $y > 0$, are well-defined but are not uniformly integrable martingales. In particular, the optimal martingale measure $\hat{\mathbb{Q}} = \mathbb{Q}(y)$ does not exist.

The proof is divided into a series of lemmas.

Lemma 6.2. *The stopping time T is finite under \mathbb{Q} and the probability measures \mathbb{P} and \mathbb{Q} are equivalent.*

Proof. In view of (6.4), we only have to show that

$$\mathbb{Q}(T < \infty) = 1.$$

Indeed, by (6.3), the intensity λ is bounded below by $\gamma > 0$ and hence,

$$\mathbb{Q}(T > t) \leq \mathbb{Q}(T_2 > t) \leq e^{-\gamma t} \rightarrow 0, \quad t \rightarrow \infty.$$

□

From the construction of the model and Lemma 6.2 we deduce that $\mathbb{Q} \in \mathcal{Q}$. To show that the density process Z of \mathbb{Q} with respect to \mathbb{P} satisfies (A_p) we need the following estimate.

Lemma 6.3. *Let $0 < \epsilon < 1$ be a constant and τ be a stopping time. Then*

$$\mathbb{E}^{\mathbb{Q}}[S_T^\epsilon | \mathcal{F}_\tau] \leq S_\tau^\epsilon \leq \left(1 + \frac{\epsilon(1-\epsilon)}{2\gamma}\right) \mathbb{E}^{\mathbb{Q}}[S_T^\epsilon | \mathcal{F}_\tau].$$

Proof. We denote

$$\theta = \frac{1}{2}\epsilon(1-\epsilon)$$

and deduce that

$$S_t^\epsilon = \mathcal{E}(B)_t^\epsilon = \mathcal{E}(\epsilon B)_t e^{-\theta t}, \quad t \in [0, T].$$

In particular, S^ϵ is a \mathbb{Q} -supermartingale, and the first inequality in the statement of the lemma follows.

To verify the second inequality, we define local martingales L and M under \mathbb{Q} as

$$\begin{aligned} L_t &= \int_0^t \frac{\theta}{\lambda_r} (dN_r - \lambda_r dr), \\ M_t &= \mathcal{E}(\epsilon B)_t \mathcal{E}(L)_t, \end{aligned}$$

and observe that

$$\begin{aligned} M_t &= S_t^\epsilon, \quad t \leq T, \quad t < T_2, \\ M_T &= \left(1 + \frac{\theta}{\lambda_T}\right) S_T^\epsilon, \quad T = T_2. \end{aligned}$$

Since $\lambda \geq \gamma$, we obtain that

$$S_t^\epsilon \leq M_t \leq \left(1 + \frac{\theta}{\gamma}\right) S_t^\epsilon, \quad t \in [0, T].$$

As $S \leq 2$, we deduce that M is a bounded \mathbb{Q} -martingale and the result readily follows. \square

Lemma 6.4. *The density process Z of \mathbb{Q} with respect to \mathbb{P} satisfies (A_p) .*

Proof. Fix a stopping time τ . As $\mathbb{Q} \sim \mathbb{P}$, we have

$$\begin{aligned} \mathbb{E} \left[\left(\frac{Z_\tau}{Z_T} \right)^{\frac{1}{p-1}} \middle| \mathcal{F}_\tau \right] &= \mathbb{E}^\mathbb{Q} \left[\left(\frac{Z_\tau}{Z_T} \right)^{1+\frac{1}{p-1}} \middle| \mathcal{F}_\tau \right] = \mathbb{E}^\mathbb{Q} \left[\left(\frac{Z_\tau}{Z_T} \right)^q \middle| \mathcal{F}_\tau \right] \\ &= \mathbb{E}^\mathbb{Q} \left[\left(\frac{\tilde{Z}_T}{\tilde{Z}_\tau} \right)^q \middle| \mathcal{F}_\tau \right], \end{aligned}$$

where $\tilde{Z} = 1/Z$ is the density process of \mathbb{P} with respect to \mathbb{Q} .

Recall that

$$\tilde{Z}_T = CS_T^b,$$

for some constant $C > 0$. Since $0 < b < bq < 1$, Lemma 6.3 yields that

$$\begin{aligned} \tilde{Z}_\tau &= \mathbb{E}^\mathbb{Q} [\tilde{Z}_T | \mathcal{F}_\tau] = C \mathbb{E}^\mathbb{Q} [S_T^b | \mathcal{F}_\tau] \geq C \left(1 + \frac{b(1-b)}{2\gamma} \right)^{-1} S_\tau^b, \\ \mathbb{E}^\mathbb{Q} [\tilde{Z}_T^q | \mathcal{F}_\tau] &= C^q \mathbb{E}^\mathbb{Q} [S_T^{qb} | \mathcal{F}_\tau] \leq C^q S_\tau^{qb}, \end{aligned}$$

which implies the result. \square

We now turn our attention to the second item of Proposition 6.1. Of course, our financial market has been specially constructed in such a way that the solutions $X(x)$ and $Y(y)$ to the primal and dual problems are quite explicit.

Lemma 6.5. *In the optimal investment problem with the utility function U from (6.5), it is optimal to buy and hold stocks:*

$$X(x) = xS, \quad x > 0.$$

The dual minimizers have the form $Y(y) = y\hat{Y}$, $y > 0$, with

$$(6.6) \quad \hat{Y} = \mathcal{E}(L)Z,$$

where Z is the density process of \mathbb{Q} with respect to \mathbb{P} and

$$(6.7) \quad L_t = \int_0^t \frac{\gamma}{\lambda_r} (\lambda_r dr - dN_r), \quad t \in [0, T].$$

Proof. We verify the conditions of Lemma 2.1. For the stochastic exponential $\mathcal{E}(L)$ we obtain that

$$\mathcal{E}(L)_t = e^{\gamma t}, \quad t < T,$$

and, as $S_{T_1} = 2$, that

$$\begin{aligned}\mathcal{E}(L)_T &= e^{\gamma T} \left(1_{\{T=T_1\}} + \left(1 - \frac{\gamma}{\lambda_T} \right) 1_{\{T=T_2\}} \right) \\ &= e^{\gamma T} \left(1_{\{T=T_1\}} + \left(\frac{S_T}{2} \right)^\delta 1_{\{T=T_2\}} \right) \\ &= e^{\gamma T} \left(\frac{S_T}{2} \right)^\delta.\end{aligned}$$

Hence for \hat{Y} defined by (6.6) we have

$$\hat{Y}_T = \mathcal{E}(L)_T Z_T = C S_T^{-a} = C U'(S_T),$$

for some constant $C > 0$.

Let $X \in \mathcal{X}$. Under \mathbb{Q} , the product $X\mathcal{E}(L)$ is a local martingale, because X is a stochastic integral with respect to the Brownian motion B and $\mathcal{E}(L)$ is a purely discontinuous local martingale. It follows that $X\hat{Y} = X\mathcal{E}(L)Z$ is a non-negative local martingale (hence, a supermartingale) under \mathbb{P} . Thus,

$$\hat{Y} \in \mathcal{Y}.$$

Observe that the convex conjugate to U is given by

$$V(y) = \frac{a}{1-a} y^{-\frac{1-a}{a}}, \quad y > 0.$$

It follows that

$$V(y\hat{Y}_T) = V(y)\hat{Y}_T \hat{Y}_T^{-1/a} = V(y)C^{-1/a}\hat{Y}_T S_T$$

and therefore,

$$\mathbb{E} \left[V(y\hat{Y}_T) \right] \leq V(y)C^{-1/a} < \infty, \quad y > 0.$$

To conclude the proof we only have to show that the local martingale $S\hat{Y} = S\mathcal{E}(L)Z$ under \mathbb{P} is of class **(D)** or, equivalently, that the local martingale $S\mathcal{E}(L)$ under \mathbb{Q} is of class **(D)**. Actually, we have a stronger property:

$$\{S_\tau \mathcal{E}(L)_\tau : \tau \text{ is a stopping time}\} \text{ is bounded in } \mathbf{L}^q(\mathbb{Q}).$$

Indeed,

$$S_t \mathcal{E}(L)_t \leq S_t e^{\gamma t} \leq 2^{1-b} S_t^b e^{\gamma t}, \quad t \in [0, T],$$

and then for a stopping time τ ,

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}[(S_{\tau}\mathcal{E}(L)_{\tau})^q] &\leq 2^{q(1-b)}\mathbb{E}^{\mathbb{Q}}\left[\left(S_{\tau}^b e^{\gamma\tau}\right)^q\right] = 2^{q(1-b)}\mathbb{E}^{\mathbb{Q}}\left[\mathcal{E}(B)_{\tau}^{qb} e^{q\gamma\tau}\right] \\ &= 2^{q(1-b)}\mathbb{E}^{\mathbb{Q}}[\mathcal{E}(qbB)_{\tau}] \leq 2^{q(1-b)}.\end{aligned}$$

□

The following lemma completes the proof of the proposition.

Lemma 6.6. *For the dual minimizer \hat{Y} constructed in Lemma 6.5 we have*

$$\mathbb{E}[\hat{Y}_T] < 1.$$

Thus, \hat{Y} is not a uniformly integrable martingale.

Proof. Recall from the proof of Lemma 6.5 that for the local martingale L defined in (6.7),

$$\mathcal{E}(L)_T = e^{\gamma T} \left(\frac{S_T}{2}\right)^{\delta}.$$

Using (6.2), we deduce that

$$\mathcal{E}(L)_T = \frac{1}{2^{\delta}} e^{\gamma T} (\mathcal{E}(B)_T)^{\delta} = \frac{1}{2^{\delta}} e^{\gamma T} \mathcal{E}(\delta B)_T e^{-\frac{1}{2}\delta(1-\delta)T} \leq \frac{1}{2^{\delta}} \mathcal{E}(\delta B)_T.$$

It follows that

$$\mathbb{E}[\hat{Y}_T] = \mathbb{E}[\mathcal{E}(L)_T Z_T] = \mathbb{E}^{\mathbb{Q}}[\mathcal{E}(L)_T] \leq \frac{1}{2^{\delta}} \mathbb{E}^{\mathbb{Q}}[\mathcal{E}(\delta B)_T] \leq \frac{1}{2^{\delta}}.$$

□

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